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On Multiple Zeta Values of Even Arguments

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Abstract

For $k \leq n$, let $E(2n,k)$ be the sum of all multiple zeta values with even arguments whose weight is $2n$ and whose depth is k. Of course $E(2n,1)$ is the value $\zeta(2n)$ of the Riemann zeta function at $2n$, and it is well known that $E(2n,2) = \frac{3}{4}\zeta(2n)$. Recently Z. Shen and T. Cai gave formulas for $E(2n,3)$ and $E(2n,4)$ in terms $\zeta(2n)$ and $\zeta(2)\zeta(2n-2)$. We give two formulas for $E(2n,k)$, both valid for arbitrary $k \leq n$, one of which generalizes the Shen-Cai results; by comparing the two we obtain a Bernoulli-number identity. We also give an explicit generating function for the numbers $E(2n,k)$.

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1 Introduction and Statement of Results

For positive integers $i_1, \ldots, i_k$ with $i_1 > 1$, we define the multiple zeta value $\zeta(i_1, \ldots, i_k)$ by

$$\zeta(i_1, \ldots, i_k) = \sum_{n_1 > \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}}. \tag{1}$$

The multiple zeta value (1) is said to have weight $i_1 + \cdots + i_k$ and depth k. Many remarkable identities have been proved about these numbers, but in this note we will concentrate on the case where the i_j are even integers. Let $E(2n, k)$ be the sum of all the multiple zeta values of even-integer arguments having weight $2n$ and depth k, i.e.,

$$E(2n, k) = \sum_{\substack{i_1, \ldots, i_k \text{ even} \\ i_1 + \cdots + i_k = 2n}} \zeta(i_1, \ldots, i_k).$$

Of course

$$E(2n, 1) = \zeta(2n) = \frac{(-1)^{n-1} B_{2n} (2\pi)^{2n}}{2(2n)!}, \tag{2}$$

where B_{2n} is the $2n$th Bernoulli number, by the classical formula of Euler. Euler also studied double zeta values (i.e., multiple zeta values of depth 2) and in his paper [2] gave two identities which read

$$\sum_{i=2}^{2n-1} (-1)^i \zeta(i, 2n - i) = \frac{1}{2} \zeta(2n)$$

$$\sum_{i=2}^{2n-1} \zeta(i, 2n - i) = \zeta(2n)$$

in modern notation. From these it follows that

$$E(2n, 2) = \frac{3}{4} \zeta(2n),$$

though Gangl, Kaneko and Zagier [3] seem to be the first to have pointed it out in print. Recently Shen and Cai [10] proved the formulas

$$E(2n, 3) = \frac{5}{8} \zeta(2n) - \frac{1}{4} \zeta(2) \zeta(2n - 2), \ n \geq 3 \tag{3}$$

$$E(2n, 4) = \frac{35}{64} \zeta(2n) - \frac{5}{16} \zeta(2) \zeta(2n - 2), \ n \geq 4. \tag{4}$$

Identity (3) was also proved by Machide [9] using a different method.

This begs the question whether there is a general formula of this type for $E(2n, k)$. The pattern

$$\frac{3}{4}, \quad \frac{3}{4}\cdot\frac{5}{6} = \frac{5}{8}, \quad \frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8} = \frac{35}{64}$$

of the leading coefficients makes one curious. In fact, the general result is as follows.

Theorem 1. *For $k \le n$,*

$$E(2n,k) = \frac{1}{2^{2(k-1)}}\binom{2k-1}{k}\zeta(2n)$$
$$- \sum_{j=1}^{\lfloor\frac{k-1}{2}\rfloor} \frac{1}{2^{2k-3}(2j+1)B_{2j}}\binom{2k-2j-1}{k}\zeta(2j)\zeta(2n-2j).$$

The next two cases after (4) are

$$\begin{aligned} E(2n,5) &= \frac{63}{128}\zeta(2n) - \frac{21}{64}\zeta(2)\zeta(2n-2) + \frac{3}{64}\zeta(4)\zeta(2n-4) \\ E(2n,6) &= \frac{231}{512}\zeta(2n) - \frac{21}{64}\zeta(2)\zeta(2n-2) + \frac{21}{256}\zeta(4)\zeta(2n-4). \end{aligned}$$

We prove Theorem 1 in §3 below, using the generating function

$$F(t,s) = 1 + \sum_{n\ge k\ge 1} E(2n,k)t^n s^k.$$

In §2 we establish the following explicit formula.

Theorem 2.

$$F(t,s) = \frac{\sin(\pi\sqrt{1-s}\sqrt{t})}{\sqrt{1-s}\sin(\pi\sqrt{t})}.$$

Our proof uses symmetric functions. We define a homomorphism $\mathfrak{Z}$: Sym $\to \mathbf{R}$, where Sym is the algebra of symmetric functions, and a family $N_{n,k} \in$ Sym such that $\mathfrak{Z}$ sends $N_{n,k}$ to $E(2n,k)$. We then obtain a formula for the generating functions

$$\mathcal{F}(t,s) = 1 + \sum_{n\ge k\ge 1} N_{n,k}t^n s^k \in \mathrm{Sym}[[t,s]]$$

and apply $\mathfrak{Z}$ to get Theorem 2.

From the form of $\mathcal{F}(t,s)$ we show that it satisfies a partial differential equation (Proposition 1 below), which is equivalent to a recurrence for the $N_{n,k}$. From the latter we obtain a formula for the $N_{n,k}$ in terms of complete and elementary symmetric functions, to which $\mathfrak{Z}$ can be applied to give the following alternative formula for $E(2n,k)$.

Theorem 3. *For $k \leq n$,*

$$E(2n,k) = \frac{(-1)^{n-k-1}\pi^{2n}}{(2n+1)!} \sum_{i=0}^{n-k} \binom{n-i}{k} \binom{2n+1}{2i} 2(2^{2i-1}-1) B_{2i}.$$

Note that the sum given by Theorem 3 has $n-k+1$ terms, while that given by Theorem 1 has $\lfloor \frac{k-1}{2} \rfloor + 1$ terms. Yet another explicit formula for $E(2n,k)$ can be obtained by setting $d=1$ in Theorem 7.1 of Komori, Matsumoto and Tsumura [7]. That formula expresses $E(2n,k)$ as a sum over partitions of k, and it is not immediately clear how it relates to our two formulas.

Comparison of Theorems 1 and 3 establishes the following Bernoulli-number identity.

Theorem 4. *For $k \leq n$,*

$$\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{2k-2i-1}{k} \binom{2n+1}{2i+1} B_{2n-2i} = (-1)^k 2^{2k-2n} \sum_{i=0}^{n-k} \binom{n-i}{k} \binom{2n+1}{2i} (2^{2i-1}-1) B_{2i}.$$

It is interesting to contrast this result with the Gessel-Viennot identity (see [1, Theorem 4.2]) valid on the complementary range:

$$\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{2k-2i-1}{k} \binom{2n+1}{2i+1} B_{2n-2i} = \frac{2n+1}{2} \binom{2k-2n}{k}, \quad k > n. \tag{5}$$

Note that the right-hand side of equation (5) is zero unless $k \geq 2n$.

2 Symmetric Functions

We think of Sym as the subring of $\mathbf{Q}[[x_1, x_2, \ldots]]$ consisting of those formal power series of bounded degree that are invariant under permutations of the x_i. A useful reference is the first chapter of Macdonald [8]. We denote the elementary, complete, and power-sum symmetric functions of degree i by e_i, h_i, and p_i respectively. They have associated generating functions

$$\begin{aligned}
E(t) &= \sum_{j=0}^{\infty} e_j t^j = \prod_{i=1}^{\infty}(1 + tx_i) \\
H(t) &= \sum_{j=0}^{\infty} h_j t^j = \prod_{i=1}^{\infty} \frac{1}{1 - tx_i} = E(-t)^{-1} \\
P(t) &= \sum_{j=1}^{\infty} p_j t^{j-1} = \sum_{i=1}^{\infty} \frac{x_i}{1 - tx_i} = \frac{H'(t)}{H(t)}.
\end{aligned}$$

As explained in [5] and in greater detail in [6], there is a homomorphism $\zeta : \mathrm{Sym}^0 \to \mathbf{R}$, where Sym^0 is the subalgebra of Sym generated by $p_2, p_3, p_4, \ldots$, such that $\zeta(p_i)$ is the value $\zeta(i)$ of the Riemann zeta function at i, for $i \geq 2$ (in [5, 6] this homomorphism is extended to all of Sym, but we do not need the extension here). Let $\mathcal{D} : \mathrm{Sym} \to \mathrm{Sym}$ be the degree-doubling map that sends x_i to x_i^2. Then $\mathcal{D}(\mathrm{Sym}) \subset \mathrm{Sym}^0$, so the composition $\mathfrak{Z} = \zeta\mathcal{D}$ is defined on all of Sym. (Alternatively, we can simply think of $\mathfrak{Z}$ as sending x_i to $1/i^2$: see [8, Ch. I, §2, ex. 21].) Note that $\mathfrak{Z}(p_i) = \zeta(2i) \in \mathbf{R}$. Further, $\mathfrak{Z}$ sends the monomial symmetric function $m_{i_1,\ldots,i_k}$ to the symmetrized sum of multiple zeta values

$$\frac{1}{|\operatorname{Iso}(i_1, \ldots, i_k)|} \sum_{\sigma \in S_k} \zeta(2i_{\sigma(1)}, 2i_{\sigma(2)}, \ldots, 2i_{\sigma(k)}),$$

where S_k is the symmetric group on k letters and $\operatorname{Iso}(i_1, \ldots, i_k)$ is the subgroup of S_k that fixes $(i_1, \ldots, i_k)$ under the obvious action.

Now let $N_{n,k}$ be the sum of all the monomial symmetric functions corresponding to partitions of n having length k. Of course $N_{n,k} = 0$ unless $k \leq n$, and $N_{k,k} = e_k$. Then $\mathfrak{Z}$ sends $N_{n,k}$ to $E(2n, k)$. Also, if we define (as in the introduction)

$$\mathcal{F}(t, s) = 1 + \sum_{n \geq k \geq 1} N_{n,k} t^n s^k,$$

then $\mathfrak{Z}$ sends $\mathcal{F}(t,s)$ to the generating function $F(t,s)$. We have the following simple description of $\mathcal{F}(t,s)$.

Lemma 1. *$\mathcal{F}(t,s) = E((s-1)t)H(t)$.*

Proof. Evidently $\mathcal{F}(t,s)$ has the formal factorization

$$\prod_{i=1}^{\infty}(1 + stx_i + st^2x_i^2 + \cdots) = \prod_{i=1}^{\infty}\frac{1+(s-1)tx_i}{1-tx_i} = E((s-1)t)H(t).$$

$\square$

Proof of Theorem 2. Using the well-known formula for $\zeta(2,2,\ldots,2)$ [4, Cor. 2.3],

$$\mathfrak{Z}(e_n) = \zeta(\underbrace{2,2,\ldots,2}_{n}) = \frac{\pi^{2n}}{(2n+1)!}. \tag{6}$$

Hence

$$\mathfrak{Z}(E(t)) = \frac{\sinh(\pi\sqrt{t})}{\pi\sqrt{t}},$$

and the image of $H(t) = E(-t)^{-1}$ is

$$\mathfrak{Z}(H(t)) = \frac{\pi\sqrt{-t}}{\sinh(\pi\sqrt{-t})} = \frac{\pi\sqrt{t}}{\sin(\pi\sqrt{t})}.$$

Thus from Lemma 1 $F(t,s) = \mathfrak{Z}(\mathcal{F}(t,s))$ is

$$\mathfrak{Z}(E((s-1)t)H(t)) = \frac{\sinh(\pi\sqrt{(s-1)t})}{\pi\sqrt{(s-1)t}}\frac{\pi\sqrt{t}}{\sin(\pi\sqrt{t})} = \frac{\sin(\pi\sqrt{(1-s)t})}{\sqrt{1-s}\sin(\pi\sqrt{t})}.$$

$\square$

Taking limits as $s \to 1$ in Theorem 2, we obtain

$$F(t,1) = \frac{\pi\sqrt{t}}{\sin\pi\sqrt{t}}$$

and so, taking the coefficient of t^n, the following result.

Corollary 1. *For all $n \geq 1$,*

$$\sum_{k=1}^{n} E(2n,k) = \frac{2(2^{2n-1}-1)(-1)^{n-1}B_{2n}\pi^{2n}}{(2n)!}.$$

Another consequence of Lemma 1 is the following partial differential equation.

Proposition 1.

$$t\frac{\partial \mathcal{F}}{\partial t}(t,s) + (1-s)\frac{\partial \mathcal{F}}{\partial s}(t,s) = tP(t)\mathcal{F}(t,s).$$

Proof. From Lemma 1 we have

$$\begin{aligned}\frac{\partial \mathcal{F}}{\partial t}(t,s) &= (s-1)E'((s-1)t)H(t) + E((s-1)t)H'(t)\\ \frac{\partial \mathcal{F}}{\partial s}(t,s) &= tE'((s-1)t)H(t)\end{aligned}$$

from which the conclusion follows. □

Now examine the coefficient of $t^n s^k$ in Proposition 1 to get the following.

Proposition 2. *For $n \geq k+1$,*

$$p_1 N_{n-1,k} + p_2 N_{n-2,k} + \cdots + p_{n-k} N_{k,k} = (n-k)N_{n,k} + (k+1)N_{n,k+1}.$$

It is also possible to prove this result directly via a counting argument like that used to prove the lemma of [6, p. 16].

The preceding result allows us to write $N_{n,k}$ explicitly in terms of complete and elementary symmetric functions as follows.

Lemma 2. *For $r \geq 0$,*

$$N_{k+r,k} = \sum_{i=0}^{r} (-1)^i \binom{k+i}{i} h_{r-i} e_{k+i}.$$

Proof. We use induction on r, the result being evident for $r = 0$. Proposition 2 gives

$$\sum_{i=1}^{r+1} p_i N_{k+r+1-i,k} = (r+1)N_{k+r+1,k} + (k+1)N_{k+r+1,k+1},$$

which after application of the induction hypothesis becomes

$$\sum_{i=1}^{r+1} \sum_{j=0}^{r+1-j} (-1)^j p_i \binom{k+j}{j} h_{r+1-i-j} N_{k+j,k+j} =$$
$$(r+1)N_{k+r+1,k} + (k+1) \sum_{j=0}^{r} \binom{k+1+j}{j} h_{r-j} N_{k+1+j,k+1+j}.$$

The latter equation can be rewritten

$$\sum_{j=0}^{r} (-1)^j \binom{k+j}{j} N_{k+j,k+j} \sum_{i=1}^{r+1-j} p_i h_{r+1-i-j} =$$
$$(r+1)N_{k+r+1,k} - (k+1) \sum_{j=1}^{r+1} (-1)^j \binom{k+j}{j-1} h_{r+1-j} N_{k+j,k+j}.$$

Now the inner sum on the left-hand side is $(r+1-j)h_{r+1-j}$ by the recurrence relating the complete and power-sum symmetric functions, so we have

$$(r+1)N_{k+r+1,k} - (r+1)N_{k,k}h_{r+1} =$$
$$\sum_{j=1}^{r+1} (-1)^j h_{r+1-j} N_{k+j,k+j} \left((r+1-j)\binom{k+j}{j} + (k+1)\binom{k+j}{j-1} \right),$$

and the conclusion follows after the observation that $(k+1)\binom{k+j}{j-1} = j\binom{k+j}{j}$. □

Proof of Theorem 3. Rewrite Lemma 2 in the form

$$N_{n,k} = \sum_{i=0}^{n-k} \binom{n-i}{k} (-1)^{n-k-i} h_i e_{n-i}$$

and apply the homomorphism $\mathfrak{Z}$, using equation (6) and

$$\mathfrak{Z}(h_i) = \frac{2(2^{2i-1}-1)(-1)^{i-1} B_{2i} \pi^{2i}}{(2i)!}.$$

□

3 Proof of Theorems 1 and 4

From the introduction we recall the statement of Theorem 1:

$$E(2n,k) = \frac{1}{2^{2(k-1)}}\binom{2k-1}{k}\zeta(2n) - \sum_{j=1}^{\lfloor\frac{k-1}{2}\rfloor}\frac{1}{2^{2k-3}(2j+1)B_{2j}}\binom{2k-2j-1}{k}\zeta(2j)\zeta(2n-2j).$$

We note that Euler's formula (2) can be used to write the result in the alternative form

$$E(2n,k) = \sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor}\frac{(-1)^j\pi^{2j}\zeta(2n-2j)}{2^{2k-2j-2}(2j+1)!}\binom{2k-2j-1}{k} \tag{7}$$

which avoids mention of Bernoulli numbers.

We now expand out the generating function $F(t,s)$. We have

$$F(t,s) = \frac{1}{\sqrt{1-s}\sin\pi\sqrt{t}}\sin(\pi\sqrt{t}\sqrt{1-s}) = \frac{\pi\sqrt{t}}{\sin\pi\sqrt{t}}\sum_{j=0}^{\infty}\frac{(-1)^j\pi^{2j}t^j(1-s)^j}{(2j+1)!} = \sum_{k=0}^{\infty}s^kG_k(t),$$

where

$$G_k(t) = (-1)^k\frac{\pi\sqrt{t}}{\sin\pi\sqrt{t}}\sum_{j\geq k}\frac{(-1)^j\pi^{2j}t^j}{(2j+1)!}\binom{j}{k}. \tag{8}$$

Then Theorem 1 is equivalent to the statement that

$$G_k(t) = \sum_{n\geq k}t^n\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor}\frac{(-1)^j\pi^{2j}\zeta(2n-2j)}{2^{2k-2j-2}(2j+1)!}\binom{2k-2j-1}{k}$$

for all k. We can write the latter sum as

$$\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-2}(2j+1)!}\binom{2k-2j-1}{k}\sum_{n\geq j+1}\zeta(2n-2j)t^{n-j}-$$
$$\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-2}(2j+1)!}\binom{2k-2j-1}{k}\sum_{n=j+1}^{k-1}\zeta(2n-2j)t^{n-j}=$$
$$\frac{1}{2}(1-\pi\sqrt{t}\cot\pi\sqrt{t})\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-2}(2j+1)!}\binom{2k-2j-1}{k}-$$
$$\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-2}(2j+1)!}\binom{2k-2j-1}{k}\sum_{n=j+1}^{k-1}\zeta(2n-2j)t^{n-j}, \quad (9)$$

where we have used the generating function

$$\frac{1}{2}(1-\pi\sqrt{t}\cot\pi\sqrt{t})=\sum_{i=1}^{\infty}\zeta(2i)t^i.$$

Note that the last sum in (9) is a polynomial that cancels exactly those terms in

$$\frac{1}{2}(1-\pi\sqrt{t}\cot\pi\sqrt{t})\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-2}(2j+1)!}\binom{2k-2j-1}{k} \quad (10)$$

of degree less than k. Thus, to prove Theorem 1 it suffices to show that

$$G_k(t) = \text{terms of degree} \geq k \text{ in expression (10)}.$$

From equation (8) it is evident that

$$G_k(t)=\frac{\pi\sqrt{t}}{\sin\pi\sqrt{t}}\cdot\frac{(-t)^k}{k!}\cdot\frac{d^k}{dt^k}\left(\frac{\sin\pi\sqrt{t}}{\pi\sqrt{t}}\right). \quad (11)$$

We use this to obtain an explicit formula for $G_k(t)$.

Lemma 3. *For $k\geq 0$,*

$$G_k(t)=P_k(\pi^2 t)\pi\sqrt{t}\cot\pi\sqrt{t}+Q_k(\pi^2 t),$$

where P_k, Q_k are polynomials defined by

$$P_k(x) = -\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor} \frac{(-4x)^j}{2^{2k-1}(2j+1)!}\binom{2k-2j-1}{k}$$
$$Q_k(x) = \sum_{j=0}^{\lfloor\frac{k}{2}\rfloor} \frac{(-4x)^j}{2^{2k}(2j)!}\binom{2k-2j}{k}.$$

Proof. In view of equation (11), the conclusion is equivalent to

$$f^{(k)}(t) = (-1)^k k! t^{-k} P_k(\pi^2 t)\cos\pi\sqrt{t} + (-1)^k k! t^{-k} Q_k(\pi^2 t) f(t),$$

where $f(t) = \sin\pi\sqrt{t}/\pi\sqrt{t}$. Differentiating, one sees that the polynomials P_k and Q_k are determined by the recurrence

$$(k+1)P_{k+1}(x) = kP_k(x) - xP_k'(x) - \frac{1}{2}Q_k(x)$$
$$(k+1)Q_{k+1}(x) = \frac{2k+1}{2}Q_k(x) - xQ_k'(x) + \frac{x}{2}P_k(x)$$

together with the initial conditions $P_0(x) = 0$, $Q_0(x) = 1$. The recurrence and initial conditions are satisfied by the explicit formulas above. □

Proof of Theorem 1. Using Lemma 3, we have

$$G_k(t) = -\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-1}(2j+1)!}\binom{2k-2j-1}{k}\pi\sqrt{t}\cot\pi\sqrt{t}$$
$$+\sum_{j=0}^{\lfloor\frac{k}{2}\rfloor} \frac{(-4\pi^2 t)^j}{2^{2k}(2j)!}\binom{2k-2j}{k} =$$
$$\frac{1}{2}(1-\pi\sqrt{t}\cot\pi\sqrt{t})\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-2}(2j+1)!}\binom{2k-2j-1}{k}$$
$$+ \text{terms of degree} < k,$$

and this completes the proof. □

Proof of Theorem 4. Using Theorem 1 in the form of equation (7), eliminate $\zeta(2n-2j)$ using Euler's formula (2) and then compare with Theorem 3 to get

$$\sum_{j=0}^{\lfloor\frac{k-1}{2}\rfloor} \frac{(-1)^{n-1}\pi^{2n}B_{2n-2j}}{2^{2k-2n-1}(2n-2j)!(2j+1)!}\binom{2k-2j-1}{k} =$$
$$\frac{(-1)^{n-k-1}\pi^{2n}}{(2n+1)!}\sum_{i=0}^{n-k}\binom{n-i}{k}\binom{2n+1}{2i}2(2^{2i-1}-1)B_{2i}.$$

Now multiply both sides by $(-1)^{n-1}2^{2k-2n-1}\pi^{-2n}(2n+1)!$ and rewrite the factorials on the left-hand side as a binomial coefficient. □

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